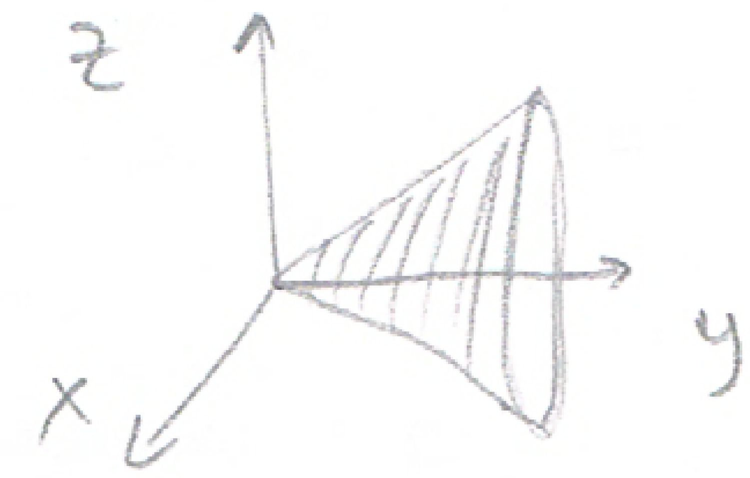


Math 2E Last Quiz Morning - June 2nd

Please write your name and ID on the front.

Show all of your work, and simplify all your answers. *There is a question on the back side.

1. Compute $\iint_S y^2 z^2 \, dS$ where S is the part of the cone $y = \sqrt{x^2+z^2}$ from $0 \le y \le 5$.

Hints: Treat this surface as a graph of a function (it may case computations). Or alternatively, a parameterization of this cone is $x = y\sin\theta,\ y = y,\ z = y\cos\theta$ with $y \in [0,5]$, $\theta \in [0, 2\pi]$.

As Graph: Note cone's shadow is $x^2+z^2 \le 25$ in xz-plane.

$$\hookrightarrow \iint_S y^2 z^2 \, dS = \iint_{x^2+z^2\le 25} z^2(x^2+z^2)\sqrt{1+\frac{(2x)^2+(2z)^2}{\left(2\sqrt{x^2+z^2}\right)^2}}\,dx\,dz \quad +2 \quad +2$$

$$= \iint_{x^2+z^2\le 25} z^2(x^2+z^2)\sqrt{1+\frac{4x^2+4z^2}{4x^2+4z^2}}\,dx\,dz$$

just $\sqrt{2}$. +1

$$\overset{\text{polar}}{=} \int_{\theta=0}^{2\pi}\int_{r=0}^{5} r^4\cos^2\theta \cdot \sqrt{2} \cdot r\,dr\,d\theta \quad +2$$

$$= \sqrt{2}\cdot \frac{r^6}{6}\Big|_0^5 \cdot \int_0^{2\pi} \frac{1}{2}(1+\cancel{\cos 2\theta})\,d\theta$$

vanishes by periodicity

$$= \sqrt{2}\cdot\frac{2\pi}{2}\cdot\frac{5^6}{6} = \boxed{\frac{5^6\pi\sqrt{2}}{6}} \quad +3$$

OR For just the 7 pts setup,

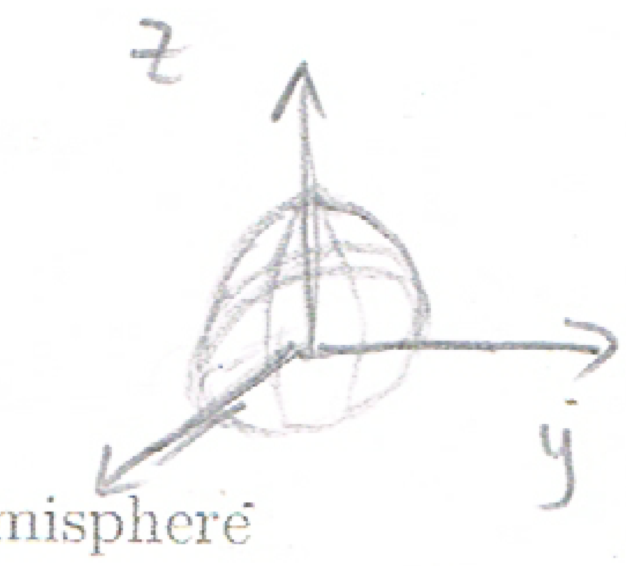
As parameterization: $\vec{r}_y = \langle \sin\theta, 1, \cos\theta\rangle$ +1, $\vec{r}_\theta = \langle y\cos\theta, 0, -y\sin\theta\rangle$ +1

$$\hookrightarrow \vec{r}_y \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta & 1 & \cos\theta \\ y\cos\theta & 0 & -y\sin\theta \end{vmatrix} = -y\sin\theta\,\hat{i} + y\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}\hat{j} - y\cos\theta\,\hat{k} \quad +2$$

$$\text{so } |\vec{r}_y\times\vec{r}_\theta| = \sqrt{y^2(\sin^2\theta+\cos^2\theta)+y^2} = \boxed{\sqrt{2}\cdot y} \quad +1$$

$$\Rightarrow \text{Integral is } \int_{y=0}^{5}\int_{\theta=0}^{2\pi} y^2\cdot y^2\cos^2\theta\cdot\sqrt{2}y\cdot d\theta\,dy \quad +2$$

$$\boxed{\text{SAME INTEGRAL}} \longrightarrow = \int_{y=0}^{5}\int_{\theta=0}^{2\pi}\sqrt{2}\,y^5\cos^2\theta\,d\theta\,dy = \boxed{\frac{5^6\pi\sqrt{2}}{6}}$$



2. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = < y, \ -x, \ 2z >$ and S consists of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ and the disk $x^2 + y^2 \leq 4$ at $z = 0$.
Assume positive orientation - the normal points outwards.
Hint: Break up the surface into two pieces, the hemisphere and the disk.

(Let disk ~ S_2
hemisphere ~ S_1.)

$\boxed{S_2}$: Is The graph of $z=0$, so $\vec{n} = <0,0,1>$, already unitized! $= \hat{n}$ +1

Then, $\vec{F} \cdot \hat{n} = <y, -x, 2z> \cdot <0,0,1> = 2z$. +1

So $\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot \hat{n}\, dS = \iint_{S_2} 2z\, dS \underset{\text{Graph of } z=0}{=} \iint_{S_2} 0\, dS = \boxed{0}$ +1

$\boxed{S_1}$: Here, $\vec{n} = \nabla(x^2+y^2+z^2-4) = <2x, 2y, 2z>$, indeed it points outwards, ✓

so $\hat{n} = \dfrac{2<x,y,z>}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{<x,y,z>}{\sqrt{x^2+y^2+z^2}} = \dfrac{<x,y,z>}{2}$ on the hemisphere. +2

↳ $\vec{F} \cdot \hat{n} = <y, -x, 2z> \cdot \dfrac{<x,y,z>}{2} = \dfrac{xy - yx + 2z^2}{2} = z^2$. +1

So, $\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \iint_{S_1} z^2\, dS$

dS sphere.

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 4\cos^2\phi \cdot 4\sin\phi \, d\phi \, d\theta$$ +1 +1

$u = \cos\phi$
$du = -\sin\phi\, d\phi$

$\overset{\theta\text{-indep}}{=} 16 \cdot 2\pi \cdot \int_1^0 -u^2\, du$; Flip Bounds b/c of $(-)$,

$= \dfrac{32\pi}{3} u^3 \Big|_0^1 = \boxed{\dfrac{32\pi}{3}}$ +1

Thus, $\iint_S = \iint_{S_1} + \iint_{S_2} \Rightarrow$ Is $0 + \dfrac{32\pi}{3} = \boxed{\dfrac{32\pi}{3}}$

+1